

# THE EMBEDDED RESOLUTION OF $f(x, y) + z^2 : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$

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## 1. INTRODUCTION

1.1. The goal of the present paper is the presentation of an “embedded resolution” of  $\{f(x, y) + z^2 = 0\}, 0 \subset (\mathbb{C}^3, 0)$  using the method of Jung. In the first part of the introduction, we present the terminology and the strategy of the paper.

Let  $(Y, 0)$  be the germ of an analytic space. We say that a proper, surjective analytic map  $\phi : \tilde{Y} \rightarrow U$  is a resolution of  $(Y, 0)$  (where  $U$  is a small representative of  $(Y, 0)$ ), if  $\tilde{Y}$  is smooth,  $\phi^{-1}(U - \text{Sing } Y)$  is dense in  $\tilde{Y}$ , and  $\phi^{-1}(U - \text{Sing } Y) \rightarrow U - \text{Sing } Y$  is an isomorphism.

More generally, if  $(Y, 0)$  is a local divisor in  $(\mathbb{C}^n, 0)$ , we say that  $\phi : \tilde{X} \rightarrow U$  is an embedded resolution of the pair  $(Y, 0) \subset (\mathbb{C}^n, 0)$  (where again  $U$  is a small representative of  $(\mathbb{C}^n, 0)$ ), if  $\tilde{X}$  is smooth,  $\phi$  is an isomorphism above  $U - \text{Sing } Y$ , and  $\phi^{-1}(Y)$  is a normal crossing divisor in  $\tilde{X}$ .

Jung’s strategy ([3], see also [5], [6]) gives a recipe how one can reduce the construction of the resolution of the  $n$ -dimensional space  $(Y, 0)$  to the case of the construction of an embedded resolution of (the “lower dimensional case”)  $(\Delta, 0) \subset (\mathbb{C}^n, 0)$  and the resolution of the so-called quasi-ordinary singularities of dimension  $n$ . (For quasi-ordinary singularities, see [5], [6]). Indeed, consider a projection  $p : (Y, 0) \rightarrow (\mathbb{C}^n, 0)$  with finite fibers, whose reduced discriminant locus is  $(\Delta, 0)$ . Then resolve the pair  $(\Delta, 0) \subset (\mathbb{C}^n, 0)$  by  $\phi : Z \rightarrow U \subset \mathbb{C}^n$  and pull back  $p$  to  $p' : Y' \rightarrow Z$ . Then  $Z$  is smooth and the branch locus of  $p'$  is a normal crossing divisor, hence  $Y'$  has only quasi-ordinary singularities. Resolving these singularities we obtain  $\tilde{Y}$ .

For example, consider the hypersurface singularity  $(Y, 0) = (\{f(x, y) + z^2 = 0\}, 0)$ , where  $f$  is an isolated plane curve singularity. Then  $p : (Y, 0) \rightarrow (\mathbb{C}^2, 0)$ , given by  $(x, y, z) \mapsto (x, y)$  is a double covering with branch locus  $(\{f = 0\}, 0) \subset (\mathbb{C}^2, 0)$ . Using the above notations, if  $\phi : Z \rightarrow U \subset \mathbb{C}^2$  is an embedded resolution of this plane curve singularity, then  $Y'$  normalized has only Hirzebruch-Jung type singularities of type  $\{z^2 = x^m y^n\}^{norm}$ .

If we want to obtain an *embedded* resolution by a similar strategy, then we face two problems. To explain this, start with  $(Y, 0) \subset (\mathbb{C}^n, 0)$  and a projection  $p : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^{n-1}, 0)$  such that  $p|_Y$  is finite. Let  $(\Delta, 0)$  be the discriminant locus of  $p|_Y$ . Then take an embedded resolution  $\phi : Z \rightarrow U^{n-1} \subset \mathbb{C}^{n-1}$  of the pair  $(\Delta, 0) \subset (\mathbb{C}^{n-1}, 0)$  and construct by pull-back the diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi'} & U^n \subset \mathbb{C}^n \\ p' \downarrow & & \downarrow p \\ Z & \xrightarrow{\phi} & U^{n-1} \subset \mathbb{C}^{n-1} \end{array}$$

1

Then still we have to find the embedded resolution of the pair  $(\phi')^{-1}(Y) \subset X$ . Although  $(\phi')^{-1}(Y)$  has only quasi-ordinary singularities, in general we know very little about their *embedded* resolution.

Moreover, even if we solve this problem (*i.e.* we find an embedded resolution  $\tilde{X} \rightarrow X$  of these quasi-ordinary singularities), the final result of the above construction has a small beauty defect: the constructed birational map  $\tilde{X} \rightarrow U^n$  is not an isomorphism above the complement of  $\text{Sing } Y$ , but only above a smaller set, the complement of  $p^{-1}(\text{Sing } \Delta)$ .

For example, if  $(Y, 0) = (\{f(x, y) + z^2 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ , and  $p(x, y, z) = (x, y)$ , and we solve all the technical problems in the above program, then we obtain a map  $\tilde{\phi} : \tilde{X} \rightarrow U^3 \subset \mathbb{C}^3$  so that  $\tilde{\phi}^{-1}(Y)$  is a normal crossing divisor, but  $\tilde{\phi}$  is an isomorphism only above  $\mathbb{C}^3 - \{x = y = 0\}$ , not above  $\mathbb{C}^3 - \{0\}$ .

Even if this birational modification does not cover exactly the above definition of the embedded resolution, in almost all the applications it plays the role of an embedded resolution: all the invariants which can be read from an embedded resolution can be read from this modification as well.

The goal of the present paper is exactly the presentation of *this* birational modification (“embedded resolution”) of  $(\{f(x, y) + z^2 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ .

**1.2. Preliminary Remarks.** Let  $g : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$  be the germ of an analytic function. Sometimes, an embedded resolution  $\tilde{\phi} : \tilde{X} \rightarrow U$  of the pair  $(\{g = 0\}, 0) \subset (\mathbb{C}^n, 0)$  (or a birational modification as above) is called the “resolution of  $g$ ”.  $E$  denotes the exceptional divisor, and  $E = E_1 \cup \dots \cup E_s$  the decomposition of  $E$  into its irreducible components.

In our situation when  $g = f(x, y) + z^2$ , since  $\tilde{\phi}$  is an isomorphism above  $\mathbb{C}^3 - \{x = y = 0\}$ , the compact irreducible exceptional divisors are situated above the origin (*i.e.*  $\tilde{\phi}(E_i) = 0$ ), while the non-compact irreducible exceptional divisors projects via  $\tilde{\phi}$  onto the disc  $\{x = y = 0\}$ .

When we want to codify the modification  $\tilde{\phi}$ , we have to decide what kind of information we would like to extract from it. In general  $\tilde{\phi} : \tilde{X} \rightarrow U$  (or even  $E$ ) carries a lot of analytic information which is impossible to codify in any topological, numerical or combinatorial object. But if we want to study the pair  $(g^{-1}(0), 0) \subset (\mathbb{C}^n, 0)$  from a topological point of view, it is enough to record only the topological/numerical/combinatorial invariants of  $\tilde{\phi}$ . If we want to codify additional analytic information as well, then we face the problem of analytic (or algebraic) classification of varieties and vector bundles (which is basically an unsolved problem). Therefore, in general, we have to find the right compromise which is still satisfactory for our final goals.

Take, for example the case  $n = 2$  and an arbitrary embedded resolution  $\tilde{\phi}$ . All the topology is completely codified in the dual resolution graph of  $E \subset \tilde{X}$  with the decorations  $\{E_i \cdot E_i\}_i$  (the self-intersections in  $\tilde{X}$ ) and the vanishing orders  $\{m_i\}_i$  of  $g \circ \tilde{\phi}$  along  $E_i$ 's. Indeed, from this decorated graph the homeomorphism type of  $(\tilde{X}, E)$ , or of  $(g^{-1}(0), 0) \subset (\mathbb{C}^n, 0)$ , can be completely recovered by plumbing. If we want to recover the analytic type of the reduced exceptional divisor  $E$ , then the combinatorics of the graph cannot identify the position of the points  $\bigcup_{j \neq i} E_j \cap E_i$  on  $E_i$ . Therefore, in this second level, if we wish to recover the isomorphic type of  $E$  from our codification, we need also to keep in this codification the pairs  $(E_i, \bigcup_{j \neq i} E_j \cap E_i)_i$ . On the other hand, at the third level, if one wants to codify

all the analytic information about  $(\tilde{X}, E)$ , then that is a rather difficult problem: the underlying topology, in general, carries many analytic structures which are parametrized by a rather complicated moduli space.

In higher dimensions, the problem is more complicated, and in general it is not clear at all what the good, convenient levels and codifications of the resolution are.

In the situation discussed in this article, when  $g(x, y, z) = f(x, y) + z^2 : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ , it turns out that the codification of the modification of  $g$  constructed above is closely related to the codification of the embedded resolution of  $f$ . We will define the analog of the resolution graph of  $f$  for  $g$ , which will keep all the topological information about the pair  $(E \subset \tilde{X})$  up to a homeomorphism. This graph will not identify  $E$  modulo an analytic isomorphism, but the ambiguity will be very similar to the plane curve singularity case. (In other words,  $g$  carries the same amount of analytic information as  $f$ .)

Actually, our modification  $\tilde{\phi} = \phi_g$  of  $g$  will be constructed from a fixed embedded resolution  $\phi = \phi_f$  of  $f$ . If we denote the irreducible exceptional divisors of  $\phi_f$  by  $\{A_i\}_i$ , then any compact irreducible exceptional divisor  $E_k$  of  $\phi_g$  will be a (possibly non-minimal) ruled surface with natural projection  $E_k \rightarrow A_i = \mathbb{P}^1$ . All the special fibers will be situated over the intersection points  $\{A_j \cap A_i\}_{j \neq i} \subset A_i$ . Similarly, the non-compact irreducible exceptional divisors will be non-minimal disc bundles over some curves  $A_i$ . If from the resolution of  $\phi_f$  we retain the information of the position of the intersection points  $\{A_j \cap A_i\}_{j \neq i} \subset A_i$ , then the analytic type of the exceptional divisors  $E_k$  will be completely determined. If we use only the dual resolution graph of  $f$ , then in the analytic type of  $E_k$  we will have the ambiguity of the position of the special fibers. This ambiguity will disappear in the computation of any kind of numerical invariant, and in the identification of different elements (like  $K_{E_k}$  or  $N_{\tilde{X}|E_k}$ ) in  $\text{Pic}(E_k)$  for any compact  $E_k$ .

## 2. REVIEW OF RULED SURFACES OVER $\mathbb{P}^1$

For a general reference for ruled surfaces, we recommend [2].

**2.1.** Any ruled surface over a smooth curve is obtained as  $\mathbb{P}(\mathcal{E})$  of a locally free sheaf (vector bundle)  $\mathcal{E}$  of rank 2. Actually,  $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{E} \otimes \mathcal{L})$  for any line bundle  $\mathcal{L}$ . But, over  $\mathbb{P}^1$ , any  $\mathcal{E}$  can be written as  $\mathcal{E} = \mathcal{O}(a) \oplus \mathcal{O}(b)$ , hence any ruled surface over  $\mathbb{P}^1$  has the “normal form”  $X_e = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-e))$  for some integer  $e \geq 0$ . The surface  $X_e$  can be obtained by gluing two copies of  $\mathbb{C} \times \mathbb{P}^1$  (with coordinates  $(x, [u_0 : u_1])$  and  $(y, [w_0 : w_1])$  respectively) along  $\mathbb{C}^* \times \mathbb{P}^1$  by the identifications  $y = 1/x$ , and  $[w_0 : w_1] = [u_0 : x^e u_1]$ . One has a natural projection  $\pi : X_e \rightarrow \mathbb{P}^1$  (in coordinates  $(x, [u_0 : u_1]) \mapsto x$ ,  $(y, [w_0 : w_1]) \mapsto y$ ). Here  $\mathbb{P}^1 = \mathbb{C} \sqcup_{\mathbb{C}^*} \mathbb{C}$ , where the  $x$ -chart  $\mathbb{C}$  corresponds to  $\{[\alpha : \beta] \mid \alpha \neq 0\}$  with  $x = \beta/\alpha$ .

An automorphism  $\phi : X_e \rightarrow X_e$  with  $\pi \circ \phi = \pi$  is called a  $\pi$ -automorphism; their collection form the group  $\mathcal{G}(X_e, \pi)$ .

The projection  $\pi : X_e \rightarrow \mathbb{P}^1$  has two natural sections with images  $C_0$  and  $C_1$ .  $C_0$  is given by  $\{u_1 = w_1 = 0\}$  and has self intersection  $C_0^2 = -e$ ;  $C_1$  is given by  $\{u_0 = w_0 = 0\}$  and has self intersection  $C_1^2 = e$ . Obviously  $C_0 \cap C_1 = \emptyset$ .

**2.2. Facts.** *a) ([2], V. 2.3)  $\text{Pic } X_e = \mathbb{Z} \oplus \pi^* \text{Pic } \mathbb{P}^1 = \mathbb{Z} \oplus \mathbb{Z}$  is generated by  $C_0$  and an arbitrary fixed fiber  $f$  of  $\pi$ . They satisfy  $C_0 \cdot f = 1$ ,  $f^2 = 0$ .*

*b) ([2], V 2.8) The invertible sheaf  $\mathcal{O}_{X_e}(1)$  (provided by the Proj-construction of  $\mathbb{P}(\mathcal{E})$ ) satisfies  $\mathcal{O}_{X_e}(1) = \mathcal{O}_{X_e}(C_0)$ .*

2.3. If  $\text{im } \tau$  is the image of a section  $\tau : \mathbb{P}^1 \rightarrow X_e$ , then  $\text{im } \tau \equiv C_0 + nf$  in  $\text{Pic } X_e$  for some  $n$ . The condition  $\text{im } \tau \cap C_0 = \emptyset$  implies  $(C_0 + nf) \cdot C_0 = 0$ , hence  $n = e$ . In particular  $(\text{im } \tau)^2 = (C_0 + ef)^2 = e$ .

2.4. If  $e = 0$ , then  $X_0$  can be represented as  $\pi : X_0 = \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ , where  $\pi$  is the first projection and  $C_0$  and  $C_1$  are two fibers of the projection on the second factor. Obviously, the system  $(\pi : X_0 \rightarrow \mathbb{P}^1; C_0, C_1)$  is uniquely determined modulo the action of  $\mathcal{G}(X_0, \pi)$  (i.e. it does not depend on the choice of  $C_0$  and  $C_1$ ).

2.5. Assume that  $e > 0$ . In the sequel we will prove that the last sentence of (2.4) is valid in this case as well. First notice that the section  $C_0$  is uniquely determined by the condition  $C_0^2 = -e$  (cf. [2], V 2.11.3). Therefore, Fact 2.2 implies that any  $\phi \in \mathcal{G}(X_e, \pi)$  keeps  $C_0$ ,  $\mathcal{O}_{X_e}(1)$ ,  $\text{Pic } X_e$  invariant.

By [2], V 2.6, there is a one-to-one correspondence between sections  $\tau : \mathbb{P}^1 \rightarrow X_e$  of  $\pi$  and surjections  $\mathcal{O} \oplus \mathcal{O}(-e) \rightarrow \mathcal{L}$ , where  $\mathcal{L} \in \text{Pic } \mathbb{P}^1$ , given by  $\mathcal{L} = \tau^* \mathcal{O}_{X_e}(1)$ . Under this correspondence,  $C_0$  corresponds to the second projection  $pr_2 : \mathcal{O} \oplus \mathcal{O}(-e) \rightarrow \mathcal{O}(-e)$  and  $C_1$  to the first projection  $pr_1 : \mathcal{O} \oplus \mathcal{O}(-e) \rightarrow \mathcal{O}$ .

Notice that it is possible to construct sections  $\tau : \mathbb{P}^1 \rightarrow X_e$  with image  $C'_1 := \tau(\mathbb{P}^1)$ , not identical to  $C_1$ , but satisfying the same numerical properties as  $C_1$ :  $C'_1 \cap C_0 = \emptyset$  and  $C'^2_1 = e$ . Nevertheless, one has:

**2.6. Proposition.** *Fix  $e > 0$  and consider a ruled surface  $\pi : X_e \rightarrow \mathbb{P}^1$  and  $C_0, C_1 \subset X_e$  as above. Then take an arbitrary section  $\tau$  with  $\text{im } \tau = C'_1$  satisfying  $C'_1 \cap C_0 = \emptyset$  (hence also  $C'_1 = C^2_1 = e$ , cf. 2.3). Then there exists  $\phi \in \mathcal{G}(X_e, \pi)$  such that  $\phi(C_1) = C'_1$ . In particular, the system  $(\pi : X_e \rightarrow \mathbb{P}^1; C_0, C_1)$  is uniquely determined (up to isomorphism) by the integer  $e$  and the conditions  $C^2_0 = -e$ ,  $C_0 \cap C_1 = \emptyset$  (and from the fact that  $C_0$  and  $C_1$  are images of sections of  $\pi$ ).  $C_1$  automatically satisfies  $C^2_1 = e$ .*

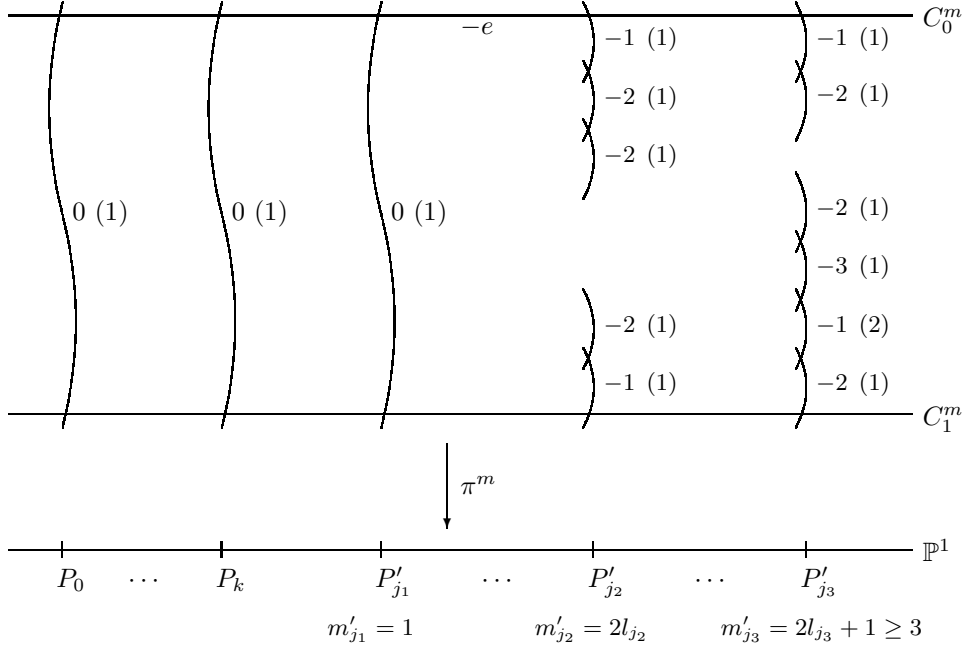
*Proof.* Let  $g : \mathcal{O} \oplus \mathcal{O}(-e) \rightarrow \mathcal{L}$  be the surjection corresponding to the section  $\tau : \mathbb{P}^1 \rightarrow X_e$  with  $\text{im } \tau = C'_1$ . Then by [2], V 2.9,  $\deg \mathcal{L} = C_0 \cdot C'_1$ , hence  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}$ . Now, consider the map  $g \oplus pr_2 : \mathcal{O} \oplus \mathcal{O}(-e) \rightarrow \mathcal{O} \oplus \mathcal{O}(-e)$ . Since  $C_0 \cap C'_1 = \emptyset$ , this is an isomorphism, which induces  $\phi$ .  $\square$

**2.7. Example.** (cf. [2] II 8.24.) Assume that two smooth surfaces  $E_1$  and  $E_0$  in the smooth three-fold  $X$  intersect each other transversally. Additionally, assume that  $E_1 \cap E_0 = C$  is isomorphic to  $\mathbb{P}^1$  and  $N_{C|E_1} = \mathcal{O}(a)$  and  $N_{C|E_0} = \mathcal{O}(b)$  with  $a \leq b$ . Let  $\tilde{X}$  be obtained by blowing up  $X$  along  $C$ , and let  $E$  be the exceptional divisor of this blowing up  $\pi$ . Then  $\pi$  induces a projection  $\pi : E \rightarrow \mathbb{P}^1$  making  $E$  a ruled surface. In fact  $E \approx X_{b-a}$ .

Let  $C_i$  be the intersection of  $E$  with the strict transform of  $E_i$  ( $i = 0, 1$ ). Then  $C_0$  and  $C_1$  are exactly the irreducible curves on  $X_{b-a}$  determined by (2.6). In particular  $N_{C_1|E} = \mathcal{O}(b-a)$  and  $N_{C_0|E} = \mathcal{O}(a-b)$ .

**2.8. Non-minimal ruled surfaces.** Now, we will fix a ruled surface  $\pi : X_e \rightarrow \mathbb{P}^1$  and a set of distinct points  $\{P_1, \dots, P_k; P'_1, \dots, P'_l\}$  on  $\mathbb{P}^1$ . Let  $Q'_j$  be the unique point on  $C_1$  with  $\pi(Q'_j) = P'_j$ .

In the sequel we will modify  $X_e$  by some blow ups using the following recipe. Above the points  $\{P_j\}_j$  we will not modify the fibers, we will only mark them. On the other hand, the fibers  $\pi^{-1}(P'_j)$  will (eventually) be modified. For this, we fix some integers  $m'_j > 0$  with  $2e = \sum_{j=1}^l m'_j$ . This also means that if  $e = 0$ , then  $l = 0$

FIGURE 1. The surface  $X_e^m$ 

and  $\{P'_1, \dots, P'_l\} = \emptyset$ , hence there is no modification. If  $e > 0$ , then we fix local coordinates  $(x_j, y_j)$  in a small neighborhood  $U_j$  of  $Q'_j$  with  $\{x_j = 0\} = C_1 \cap U_j$ , and we consider a local curve  $D = \{x_j^2 = y_j^{m'_j}\}$  in  $U_j$ . Then we will modify  $U_j$  by a minimal sequence of blow ups so that the total transform of  $D \cup C_1$  form a normal crossing divisor. Here we distinguish three cases. If  $m'_j = 1$ , then  $D = \{x_j^2 = y_j\}$  intersects  $C_1$  transversally, hence no blow up is needed. If  $m'_j$  is odd and  $m'_j \geq 3$ , then  $(m'_j + 3)/2$  blow ups are needed. Finally, if  $m'_j$  is even, then one needs  $m'_j/2$  blow ups. We do this for any  $1 \leq j \leq l$ . Notice that the sequence of blow ups (in particular, the analytic type of the new surface) is independent of the choices of the local coordinates  $(x_j, y_j)$  in  $U_j$ , depending only on the integers  $m'_j$  (and the position of the points  $\{P'_j\}_j$  in  $\mathbb{P}^1$ ). The new surface will be denoted by  $X_e^m$ . The surface  $X_e^m$  and the projection  $\pi^m : X_e^m \rightarrow \mathbb{P}^1$  (where  $\pi^m = \pi \circ$  the sequence of blow ups) looks as in Figure 1.

Here  $C_i^m$  denotes the strict transform of  $C_i$  ( $i = 0, 1$ ). The strict transform of  $\pi^{-1}(P'_j) \subset X_e$  is the unique irreducible component of  $(\pi^m)^{-1}(P'_j)$  which intersects  $C_0^m$ . The other components of  $(\pi^m)^{-1}(P'_j)$  are the new exceptional divisors. The non-positive integer near each irreducible curve denotes the self-intersection of the curve in  $X_e^m$ . The positive integer in parenthesis is the multiplicity of the corresponding irreducible component in the divisor  $\pi^*(P'_j)$ .

It is clear that

2.9.

$$\begin{cases} (C_0^m)^2 = -e \\ (C_1^m)^2 = e - \sum_{m'_j \text{ even}} m'_j/2 - \sum_{m'_j \text{ odd} > 1} (m'_j + 1)/2 \end{cases}$$

Moreover, by [2], V 3.2, one has:

$$2.10. \quad \text{Pic}(X_e^m) = \mathbb{Z}^r, \text{ where } r = 2 + \sum_{m'_j \text{ even}} m'_j/2 + \sum_{m'_j \text{ odd} > 1} (m'_j + 3)/2$$

and it is generated by  $C_0^m$ , a generic fiber  $f$ , and the new irreducible exceptional curves  $\{C'_j\}$ . The intersection matrix can be easily read from Figure 1 using  $(C_0^m)^2 = -e$ ,  $f^2 = 0$ ,  $C_0^m \cdot f = 1$ , and  $C_0^m \cdot C'_j = f \cdot C'_j = 0$  for any irreducible exceptional curve  $C'_j$ .

Notice that the isomorphism type of  $X_e^m$  is completely determined by the integer  $e$ ,  $\{m'_j\}_{j=1}^l$  and the position of the points  $\{P'_j\}$ . On the other hand, the homeomorphism type of the system  $(X_e^m, C_0^m, C_1^m, \{(\pi^m)^{-1}(P_j)\}_j, \{(\pi^m)^{-1}(P'_j)\}_j)$  is determined completely by  $e$  and  $\{m'_j\}$ , and does not depend on the choice of the points  $\{P_j\}_j$  and  $\{P'_j\}_j$ .

2.11. Any compact irreducible exceptional divisor of the birational modification which will be constructed later, associated with  $g = f(x, y) + z^2$ , will be isomorphic to some  $X_e^m$ . The irreducible components of  $C_0^m \cup C_1^m \cup (\cup_j (\pi^m)^{-1}(P_j)) \cup (\cup_j (\pi^m)^{-1}(P'_j))$  provide the intersection curves with other irreducible exceptional divisors.

On some of the irreducible exceptional divisors  $X_e^m$  we have to put one more curve: the intersection of  $X_e^m$  with the strict transform of  $\{g = 0\}$ . This discussion is postponed until (3.5).

2.12. The non-compact irreducible exceptional divisors of  $g$  will be (non-minimal) disc bundles over  $\mathbb{P}^1$ . Notice that for any  $e \in \mathbb{Z}$ , there is a unique disc bundle  $\pi : B_e \rightarrow \mathbb{P}^1$  such that the zero section  $C$  satisfies  $C^2 = e$ . Similarly as in the case of  $X_e$ , we can fix some data which codifies a sequence of blow ups with centers above the zero section of  $\pi$ . In this way, we obtain the (non-minimal) modified disc bundle  $B_e^m$  with natural projection  $B_e^m \rightarrow \mathbb{P}^1$ . These are the candidates for the non-compact irreducible exceptional divisors of our birational modification.

### 3. THE EMBEDDED RESOLUTION OF $g = f(x, y) + z^2$

3.1. First we fix some notations regarding a convenient embedded resolution graph of the isolated plane curve singularity  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ .

Let  $\phi_f : Z \rightarrow U^2$  be an embedded resolution of the pair  $(f^{-1}(0), 0) \subset (\mathbb{C}^2, 0)$ , where  $U^2$  is a small representative of  $(\mathbb{C}^2, 0)$ . We denote the irreducible exceptional divisors by  $\{A_1, \dots, A_s\}$ , the strict transforms of the irreducible components of  $\{f = 0\}$  by  $\{St_1, \dots, St_{s'}\}$ , and the collection of all these irreducible components by  $\{D_1, \dots, D_{s+s'}\}$ . It is obvious that each  $A_i$  is rational. Topologically, the pair  $\phi_f^{-1}(\{f = 0\}) \subset Z$  (or  $(\{f = 0\}, 0) \subset (\mathbb{C}^2, 0)$ ) is codified by the following data:

- a) the intersection matrix  $(A_i \cdot A_j)_{i,j}$ ;
- b) the intersections  $A_i \cdot St_j$ ;
- c) the multiplicities  $m_i(f)$  of  $f \circ \phi_f$  along  $D_i$  (actually a, b  $\Rightarrow$  c).

In the next construction, it is convenient to assume that there is no pair  $D_i$  and  $D_j$  with  $D_i \cdot D_j \neq 0$  such that both  $m_i(f)$  and  $m_j(f)$  are odd.

This situation can always be realized: indeed if  $D_i \cap D_j \neq \emptyset$  and  $m_i(f) \cdot m_j(f) \equiv 1 \pmod{2}$ , then we blow up the intersection point  $D_i \cap D_j$ . The multiplicity of the new exceptional divisor will be the even number  $m_i(f) + m_j(f)$ . Notice that there is a unique minimal resolution  $\phi_f$  with this property.

We will use the following notations as well. For a generic point  $P$  on  $A_i$ , there are local coordinates  $(u, v)$  in a small neighborhood  $U$  of  $P$  such that  $f \circ \phi_f|_U = u^{m_i(f)}$ . Similarly, if  $P = D_i \cap D_j$ , then in some local coordinates in a small neighborhood  $U$  of  $P$  one has  $f \circ \phi_f|_U = u^{m_i(f)}v^{m_j(f)}$ .

3.2. In the construction of the embedded resolution graph of  $g(x, y, z) = f(x, y) + z^2$ , we will use Jung's strategy. Consider the following diagram (already mentioned above):

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{r} & X & \xrightarrow{\phi'} & U^3 \supset g^{-1}(0) \\ & & \downarrow p' & & \downarrow p \\ & & Z & \xrightarrow{\phi_f} & U^2 \supset f^{-1}(0) \end{array}$$

We have:

- $U^3$  is a small representative (polydisc) of  $(\mathbb{C}^3, 0)$  and  $p : U^3 \rightarrow U^2$  is induced by the projection  $(x, y, z) \mapsto (x, y)$ .
- $\phi_f : Z \rightarrow U^2$  is an embedded resolution of  $(f^{-1}(0), 0) \subset (\mathbb{C}^2, 0)$  as described in (3.1).
- $p' : X \rightarrow Z$  is the pull-back of  $p : U^3 \rightarrow U^2$  via  $\phi_f$ . Notice that  $X$  is smooth. Let  $T_g := (\phi')^{-1}(g^{-1}(0))$  be the total transform of  $g^{-1}(0)$  in  $X$ . Fix a generic point  $A_i$  and small coordinate neighborhood  $U$  of  $P$  as in (3.1). Then  $(p')^{-1}(U)$  admits local coordinates  $(u, v, z)$ , where  $p'(u, v, z) = (u, v)$ , and  $T_g \subset (p')^{-1}(U)$  has equation  $u^{m_i(f)} + z^2 = 0$ . Similarly, if  $P = D_i \cap D_j$ , then  $T_g$  is given by  $u^{m_i(f)}v^{m_j(f)} + z^2 = 0$ .
- $r$  is an embedded resolution of  $T_g \subset X$  (cf. 3.4). The composed map  $\phi' \circ r$  is denoted by  $\phi_g$ . By construction  $\phi_g : \tilde{X} \rightarrow U^3$  is an “embedded resolution” of  $(g^{-1}(0), 0) \subset (\mathbb{C}^3, 0)$ , which is an isomorphism above  $U^3 - \{x = y = 0\}$ , cf. Introduction.

Now we will describe the exceptional divisor  $E$  of  $\phi_g$ .

3.3. The exceptional divisor  $E$  is a union  $E_{nc} \cup E_c$  where  $E_{nc}$  (respectively  $E_c$ ) is the union of non-compact (respectively, compact) irreducible exceptional components.  $E_{nc}$  is created in two steps: first we create the exceptional divisors of  $\phi'$ , then we modify them by some blow ups. Indeed, the resolution  $\phi_f$  gives rise to the exceptional curve  $A = \phi_f^{-1}(0)$ . This lifted, gives rise to the exceptional surfaces  $A \times D = (\phi')^{-1}(D) \subset X$ , where  $D$  is the disc  $\{x = y = 0\} \subset U^3$ . Each irreducible component of  $A \times D$  has the form  $A_j \times D$ , i.e. it is a disc bundle with trivial self-intersection of the zero section. We denote  $A_j \times D$  by  $E(A_j)$ .

The multiplicity of the function  $g$  (or of  $z$ ) along each  $E(A_j)$  is zero.

If  $A'_j = (p'|_{T_g})^{-1}(A_j) \subset T_g$ , then  $N_{A'_j|X} = \mathcal{O} \oplus \mathcal{O}(A_j^2)$ .

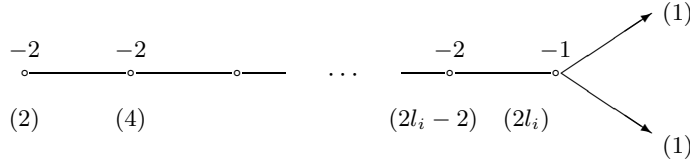
3.4. Now we describe  $r : \tilde{X} \rightarrow X$  in more detail. First, we fix an ordering of the irreducible exceptional divisors  $\{A_j\}_{1 \leq j \leq s}$  of  $\phi_f$ . By convention, if  $m_i(f)$  is even and  $m_j(f)$  is odd, then  $i < j$ . (Sometimes we say that  $A_i$  is older than  $A_j$  if  $i < j$ .) Such an ordering always exists and in general it is not unique. Different orderings provide different modifications.

Now, fix an irreducible component  $A_i$ . Fix a generic point  $P$  on  $A_i$ , let  $U$  be a small neighborhood of  $P$  as in (3.1). Then  $T_g \cap U = \{(u, v, z) : u^{m_i(f)} + z^2 = 0\}$ , where  $\{u = 0\} = E(A_i) \cap U$ .

The transversal plane curve singularity has type  $A_{m_i(f)-1}$ ; corresponding to its minimal embedded resolution, we blow up the corresponding rational curves above  $A_i$ . First we resolve this transversal singularity completely above  $A_1$ , then we continue with  $A_2$ , and so on. If  $m_i(f)$  is even, we need  $m_i(f)/2$  blow ups; if  $m_i(f) = 1$  then we need no modification, if  $m_i(f)$  is odd and  $> 1$ , then we need  $(m_i(f) + 3)/2$  blow ups.

More precisely, assume that we finished this procedure for  $A_1, \dots, A_{i-1}$ , and we want to continue with the curve  $A_i$ .

If  $m_i(f)$  is even,  $m_i(f) = 2l_i$ , then the graph of the minimal embedded resolution of  $(\{u^{2l_i} + z^2 = 0\}, 0) \subset (\mathbb{C}, 0)$  is:



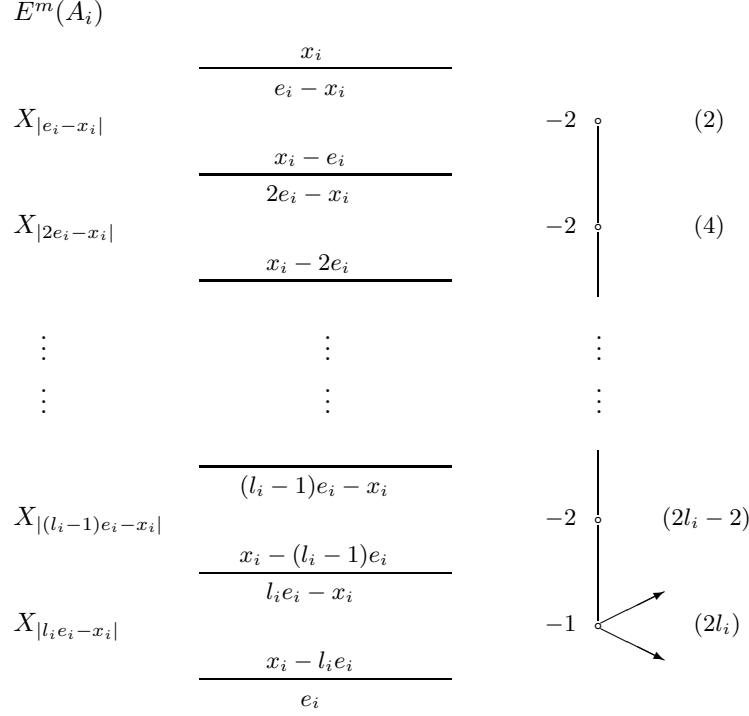
where  $(n)$  denotes the multiplicity of  $u^{m_i(f)} + z^2$  along the corresponding irreducible exceptional divisor. Corresponding to this, we make  $l_i$  blow ups along the corresponding rational curves (as axis) above  $A_i$ . Let's see what types of ruled surface we will obtain.

If  $A_j$  is older than  $A_i$ , *i.e.*  $j < i$ , then  $m_j(f)$  is automatically even, hence  $E(A_i)$  is modified  $\sum_{j < i, A_i A_j \neq 0} m_j(f)/2$  times in different infinitesimally close points. After these modifications  $E(A_i)$  becomes  $E(A_i)'$ . The strict transform of  $A_i'$  in  $E(A_i)'$  is denoted by  $A_i''$ . Then  $A_i''$  has normal bundle  $\mathcal{O}(-\sum_{j < i} m_j(f)/2) \oplus \mathcal{O}(A_i^2)$ . For simplicity, we write  $A_i^2 = e_i$  and  $-\sum_{j < i} m_j(f)/2 = x_i$ . Then we start to resolve the transversal singularity  $A_{m_i(f)-1}$  above  $A_i$ . After the sequence of blow ups along the rational curves above  $A_i$ , we obtain a tower of ruled surfaces above  $A_i$ , as it is shown in Figure 2, (*cf.* also with 2.7). In this diagram the schematic picture



denotes the ruled surface  $X_{|n|}$ , the horizontal lines denote the two distinguished curves  $C_0$  and  $C_1 \subset X_{|n|}$  with self-intersections  $\pm n$ . In Figure 2 an adjacency shows that  $X_{|n|}$  and  $X_{|m|}$  intersect each other along their distinguished curve codified by



FIGURE 2. Tower of ruled surfaces, the case  $m_i(f) = 2l_i$ 

the common horizontal line. The arrows on the graph of  $\{u^{2l_i} + z^2 = 0\}$  correspond to the strict transforms; their contribution will be discussed in (3.5). Obviously, the integers  $(n)$  denote the vanishing orders of  $g$  along the corresponding components  $X_{|n|}$ .

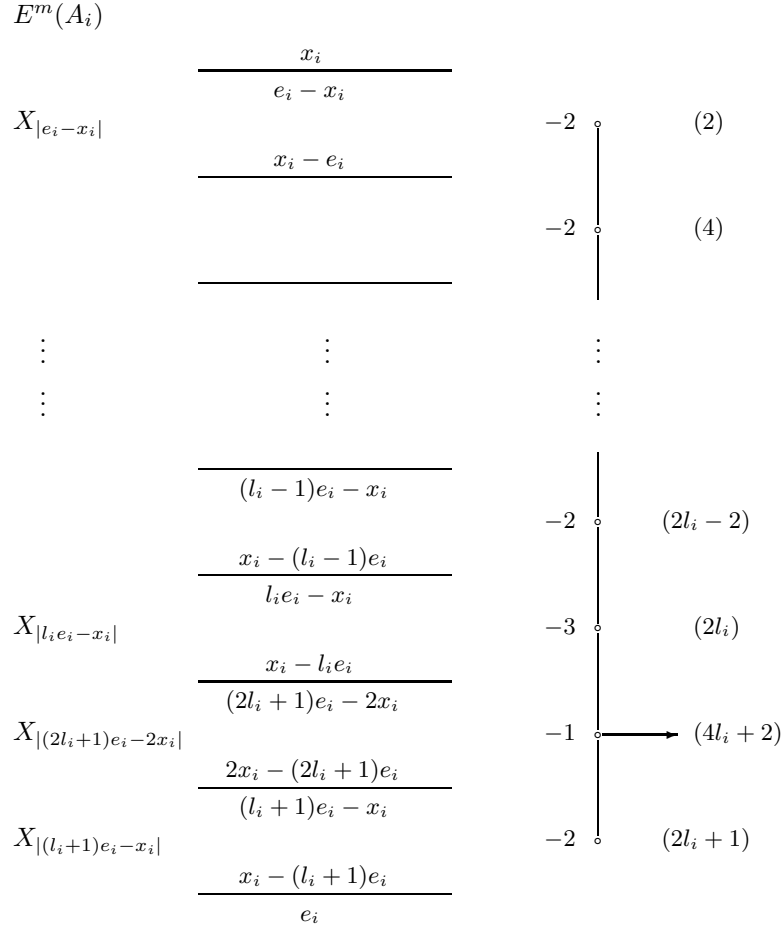
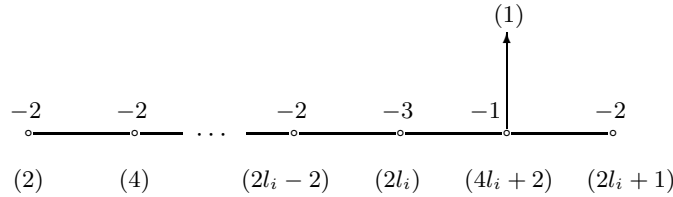
Since the divisor  $(f)$  of  $f$  on  $Z$  satisfies  $(f) \cdot A_i = 0$ , one has

$$e_i m_i(f) + \sum_{j < i, A_j A_i \neq 0} m_j(f) + \sum_{j > i, A_j A_i \neq 0} m_j(f) = 0.$$

Therefore,  $x_i - e_i l_i = \frac{1}{2} \sum_{j > i, A_j A_i \neq 0} m_j(f) \geq 0$ .

Therefore, the surface at the bottom of the tower is  $X_{x_i - e_i l_i}$ . The collection  $\{m_j(f)\}_{j > i, A_j A_i \neq 0}$  is sometimes denoted by  $\{m'_j(f)\}_j$ . The bottom horizontal line is the distinguished curve  $C_1$  of  $X_{x_i - e_i l_i}$  (since it has positive self-intersection). All the other surfaces will be unchanged by the latter modifications, but this  $X_{x_i - e_i l_i}$  will be changed by some blowing ups corresponding to the modifications of the younger neighbors  $A_j$  ( $j > i$ ) of  $A_i$ .

If  $m_i(f) = 2l_i + 1$  is odd, then the graph of the minimal resolution of  $\{u^{2l_i+1} + z^2 = 0\}$  is

FIGURE 3. Tower of ruled surfaces, the case  $m_i(f) = 2l_i + 1$ 

Since all the neighbors of  $A_i$  are older than  $A_i$  (since all of them have even multiplicity),  $N_{A'_i|X} = \mathcal{O}(x_i) \oplus \mathcal{O}(e_i)$  where  $x_i = -\sum_{i \neq j, A_i A_j \neq 0} m_j(f)/2$  and  $e_i = A_i^2$  as above. Then the tower of ruled surfaces is as in Figure 3.

The relation  $(f) \cdot A_i = 0$  now reads as

$$m_i(f)e_i + \sum_{A_j A_i \neq 0, i \neq j} m_j(f) = 0$$

*i.e.*  $(2l_i + 1)e_i - 2x_i = 0$ . Therefore, the second surface from the bottom (which will support the strict transform corresponding to the arrow) is  $X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ .

$E^m(A_j)$	(0)		$\tilde{D}$	
$X_{ e_j - x_j }$	(2)	0	-2	$E^m(A_i)$
	(4)	0	-2	
	$\vdots$			
	( $2t_j - 2$ )	0	-2	(0)
	( $2t_j$ )	-1	-1	
		-2	0	(2) $X_{ e_i - x_i }^m$
		-2	0	(4)
				$\vdots$
		-2	0	( $2l_i - 2$ )
$X_{x_j - t_j e_j}^m$		-1	0	( $2l_i$ ) $X_{x_i - l_i e_i}^m$

FIGURE 4. Gluing, the case  $m_i(f) = 2l_i$ ,  $m_j(f) = 2t_j$  and  $j < i$ 

Now we will analyze how we have to glue all these towers of surfaces. This discussion will clarify also how we have to modify  $X_{x_i - e_i l_i}$  (the case  $m_i(f)$  even) corresponding to the blowing ups of the younger neighbors.

If  $P$  is an intersection point  $A_i \cap A_j$ , then  $T_g \subset (p')^{-1}(U)$  has local equation  $u^{m_i(f)}v^{m_j(f)} + z^2 = 0$ . We distinguish two cases. In the first case  $m_i(f) = 2l_i$ ,  $m_j(f) = 2t_j$ , and we assume that the  $u$ -axis is older than the  $v$ -axis (*i.e.*  $j < i$ ). Then above  $U$ , the exceptional divisor  $E$  has the form shown as in Figure 4. In the picture, the  $\tilde{D}$  is a disc over  $D = \{x = y = 0\}$ . The vertical segments of contact between projective surfaces denote rational curves. The integers  $a|b$  denote the self-intersection numbers of this curve in the two surfaces correspondingly.

If the local equation is  $u^{m_i(f)}v^{m_j(f)} + z^2 = 0$ , where  $m_i(f) = 2l_i + 1$ ,  $m_j(f) = 2t_j$ , then automatically  $j < i$ , and  $E$  above a neighborhood of  $P$  looks as in Figure 5.

Now, it is possible to verify using a local equation of type  $u^{m_i(f)}v^{m_j(f)} + z^2 = 0$ , that in both cases after the steps described above the total transform of  $\{g = 0\}$  is

$E^m(A_j)$	(0)			$\tilde{D}$	
$X_{ e_j-x_j }$	(2)	0	-2	$E^m(A_i)$	
	(4)	0	-2		
	$\vdots$				
	$(2t_j - 2)$	0	-2		(0)
	$(2t_j)$	-1	-1		
		-2	0	(2)	$X_{ e_i-x_i }^m$
		-2	0	(4)	
				$\vdots$	
		-3	0	$(2l_i)$	
		-1	0	$(4l_i + 2)$	
		-2	0	$(2l_i + 1)$	
$X_{x_j-t_j e_j}^m$					

FIGURE 5. Gluing, the case  $m_i(f) = 2l_i + 1$ ,  $m_j(f) = 2t_j$ 

a normal crossing divisor, *i.e.* we do not have to blow up any other center, and the resolution procedure ends (*cf.* Orbanz [8].)

This ends the complete description of the exceptional set  $E$  and of all the normal bundles  $N_{E_k \cap E_{k'} | E_k}$  ( $k \neq k'$ ).

### 3.5. The intersection of the strict transform $St(g)$ of $\{g = 0\}$ with $E$ .

Corresponding to the arrows with multiplicity (1) of the embedded resolution graph of the plane curve singularity  $\{u^m + z^2 = 0\}$ , in the tower of surfaces above any  $A_i$ , there is exactly one, say  $\tilde{E}(A_i)$ , which intersects the strict transform  $St(g)$ . Let us denote this intersection by  $S_i^m = St(g) \cap \tilde{E}(A_i)$ .

First we will describe the position of  $S_i^m$  in  $\tilde{E}(A_i)$ .

If  $m_i(f)$  is odd, the  $\tilde{E}(A_i) = X_{(2l_i+1)e_i-2x_i} = X_0 = \mathbb{P}^1 \times \mathbb{P}^1$ . If by this identification  $\pi : \tilde{E}(A_i) \rightarrow A_i$  corresponds to the first projection, and  $C_0$  and  $C_1$ , the intersection curves with the neighbor ruled surfaces in the same tower correspond to two fibers of the second projection, then  $S_i^m$  provides another section of  $\pi$  with  $S_i^m \cap C_0 = S_i^m \cap C_1 = \emptyset$ . Therefore  $S_i^m$  is another fiber of the second projection. Obviously its self-intersection is  $(S_i^m)^2 = 0$  and  $S_i^m \cdot f = 1$ .

The situation is slightly more complicated if  $m_i(f) = 2l_i$  is even.

Consider that moment of the resolution procedure when we finished the construction of the tower about  $A_i$ : we just created the last ruled surface  $X_{x_i-l_i e_i}$ , but we did not start the next tower above  $A_{i+1}$ . Consider the intersection points  $\{P_0, \dots, P_k\}$  of  $A_i$  with older exceptional curves  $A_j$  ( $j < i$ ), and also all the intersection points  $\{P'_1, \dots, P'_l\}$  of  $A_i$  with all the other irreducible components of the total transform of  $\{f = 0\}$ , i.e. with the younger exceptional curves  $A_j$  ( $j > i$ ) and with the strict transforms  $St_j$ . Then, similarly as above, we denote the collection of multiplicities  $m_j(f)$  of  $f$  along the components  $A_j$  ( $A_j \cdot A_i \neq 0$ ,  $j > i$ ) and  $St_j$  ( $St_j \cdot A_i \neq 0$ ) by  $\{m'_t\}_{t=1}^l$  (such that the index corresponds to the index of the points  $P'_t$ ). Then we are in the situation of (2.8) where  $X_e = X_{x_i-l_i e_i}$ . Using the notation of (2.8), the modified surface  $X_e^m$  is exactly the surface which is obtained from  $X_{x_i-l_i e_i}$  after we finish the resolution procedure, i.e. we construct all the neighbor towers as well.

Moreover, the intersection with the strict transform can be identified as follows. Assume that the  $x$ -chart  $\mathbb{C}_x$  of  $A_i$  contains all the points  $\{P'_t\}_{t=1}^l$ . Then the intersection  $S_i$  of the strict transform of  $g$  with  $X_e$  in the  $\mathbb{C}_x \times \mathbb{P}^1$  chart is

$$\left\{ (x, [u_0 : u_1]) : u_0^2 = u_1^2 \prod_{t=1}^l (x - x_t)^{m'_t} \right\}.$$

In particular, it is uniquely determined by the pair  $(A_i, \{P'_t\}_t)$  and from the numerical data. After we blow up  $X_e$  and we obtain  $X_e^m$ , denote the strict transform of  $S_i$  in  $X_e^m$  by  $S_i^m = St(g) \cap X_e^m$ . Schematically,  $S_i^m$  is as in Figure 6 (but this is a “real picture”, which does not reflect exactly the “complex picture”). In the picture, the points  $P'_i$  with  $m'_i = 1$  correspond exactly to the intersection points of  $A_i$  with the components of the strict transform of  $\{f = 0\}$ . Above these points  $S_i^m$  intersects transversally  $C_1^m$ , and these are the only intersection points of  $S_i^m$  and  $C_1^m$ .

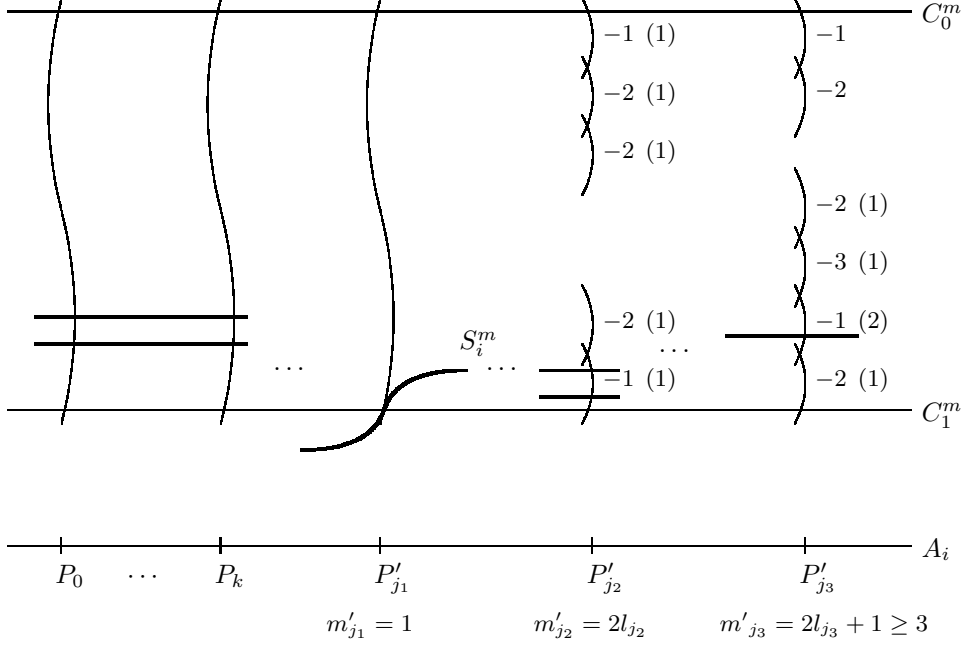
Therefore,

$$\begin{aligned} S_i^m \cdot C_1^m &= \#\{\text{strict transforms of } f = 0 \text{ supported on } A_i\} \\ &= \#\{m_j(f) = 1; D_j \cdot A_i \neq 0\} \end{aligned}$$

Since  $S_i^m \rightarrow A_i$  is a double covering with branch locus exactly over the points  $P'_j$  with  $m'_j$  odd, one has:

- $S_i^m \cdot f = 2$
- If each  $m'_j$  is even (i.e.  $A_i$  has no neighbor with odd multiplicity) then  $S_i^m$  has two disjoint irreducible components, both isomorphic to  $A_i (\approx \mathbb{P}^1)$ .
- If at least one  $m'_j$  is odd, then  $S_i^m$  is irreducible; its genus can be computed by an Euler characteristic argument (or Hurwitz’s formula, see [2] page 299):

$$\text{genus}(S_i^m) = \frac{\#\{j : m'_j \text{ odd}\} - 2}{2}.$$

FIGURE 6.  $S_i^m$ , the case  $m_i(f) = 2l_i$ 

Now we determine the self-intersection of  $S_i^m$ . The above diagram shows that  $S_i^m \cdot C_0^m = 0$  and also one can read all the intersections of  $S_i^m$  with the new exceptional divisors of  $X_e^m$ .

If  $m'_j$  is odd and  $\geq 3$ , then above  $P'_j$  (i.e. in  $(\pi^m)^{-1}(P'_j)$ ) we distinguish two irreducible curves  $F'_j$  and  $F_j$  defined by  $F'_j \cdot C_i^m = 1$ ,  $F'_j \cdot F_j = 1$ .

Notice that  $F'_j$  is the unique curve with multiplicity two in the divisor  $(\pi^m)^*(P'_j)$  and  $S_i^m$  intersects the fiber  $(\pi^m)^{-1}(P'_j)$  exactly along  $F'_j$ . Therefore  $S_i^m \cdot F'_j = 1$ . We invite the reader to verify that in  $\text{Pic}(X_e^m)$  one has

$$S_i^m = 2C_1^m + \sum F_j, \quad (\text{sum over } m'_j \text{ odd } \geq 3).$$

Therefore,

$$\begin{aligned} (S_i^m)^2 &= (2C_1^m + \sum F_j)^2 \\ &= 4(C_1^m)^2 + 4M_i - 2M_i = 4(C_i^m)^2 + 2M_i, \end{aligned}$$

where  $M_i = \#\{j : m'_j \text{ odd}, m'_j \geq 3\}$ .

If  $m_i(f)$  is even, and there is no  $D_i$  with  $D_i \cdot A_i \neq 0$  with odd multiplicity, then  $S_i^m$  has two disjoint components, each  $\approx C_1^m \approx \mathbb{P}^1$  and  $\equiv C_1^m$  in  $\text{Pic}(X_e^m)$ .

**3.6. The self-intersections  $E_k^2$ .** Consider all the compact irreducible exceptional divisors  $\{E_k\}_k$  of the resolution  $\tilde{\phi}$ . In this paragraph we determine the “self-intersections”  $E_k^2 := \mathcal{O}_{\tilde{X}}(E_k)|_{E_k} \in \text{Pic}(E_k)$ . First notice that in the previous discussions we have determined completely the divisor  $(g \circ \tilde{\phi}) = St(g) + \sum_l m_l E_l$  (where the sum is over the compact irreducible exceptional divisors). But, for any

$k$ ,  $(g \circ \tilde{\phi}) \cdot E_k = 0$  in  $\text{Pic}(E_k)$ , therefore:

$$m_k E_k^2 = (-St(g) - \sum_{l \neq k} m_l E_l) E_k$$

in  $\text{Pic}(E_k)$ . But all the intersections  $St(g)E_k$  and  $E_l E_k$  ( $l \neq k$ ) are determined above, hence  $E_k^2$  follows.

**3.7. The resolution of  $(\{f(x, y) + z^2 = 0\}, 0)$ .** Notice that  $\tilde{\phi} : \tilde{X} \rightarrow \mathbb{C}^3$  induces a map  $\tilde{\phi} : St(g) \rightarrow \{g = 0\}$ , which is a resolution of the normal surface singularity  $\{g = 0\}$ . Its exceptional curve is exactly  $\bigcup_i S_i^m$ . In this subsection we determine the self-intersections of the irreducible components of  $\bigcup_i S_i^m$ , and their combinatorics. In particular, we re-obtain the dual resolution graph of this resolution (which was known *cf.* [4], see also [7]). The details are left to the reader.

We distinguish several cases.

If  $m_i(f)$  is odd, then  $\pi : S_i^m \rightarrow A_i$  is an isomorphism, hence  $S_i^m$  is rational. Above each intersection point  $A_i \cap A_j$  we have exactly one intersection point  $S_i^m \cap S_j^m$ . The self-intersection of  $S_i^m$  in  $St(g)$  is  $e_i/2$  where  $e_i = A_i^2$ .

If  $m_i(f)$  is even, and any  $D_j$  with  $D_j \cdot A_i \neq 0$  has even multiplicity, then  $S_i^m$  has two irreducible components, each isomorphic to  $A_i = \mathbb{P}^1$ . In this case the self-intersection of each component in  $St(g)$  is  $e_i$ . Above an intersection point  $A_i \cap A_j$  we have exactly two intersection points of  $S_i^m \cap S_j^m$ .

If  $m_i(f)$  is even, and at least one of the neighbors (including the strict transform components) has odd multiplicity  $m_j(f)$ , then  $S_i^m$  is an irreducible curve with genus  $= (\lambda - 2)/2$  where  $\lambda$  is the number of odd neighbors (including the strict transform components) (*cf.* also with 3.5). Its self-intersection is  $2e_i$ . Above an intersection point  $A_i \cap D_j$  with  $m_j(f)$  even, there are two intersection points; if  $m_j(f)$  is odd, then only one intersection point of  $S_i^m \cap S_j^m$ .

This determines completely the dual resolution graph of  $\{f(x, y) + z^2 = 0\}$  from the embedded resolution graph of  $f$ . If we start with the minimal embedded resolution of  $f$  which has the additional property that there are no neighbors  $D_i$  and  $D_j$ , both with odd multiplicity, then the constructed resolution  $St(g) \rightarrow \{g = 0\}$  is exactly the *canonical* resolution of  $\{g = 0\}$  (*cf.* [4]).

The above statements about the self-intersections of the components of  $S_i^m$  in  $St(g)$  can be obtained by the following “triple point formula” as well.

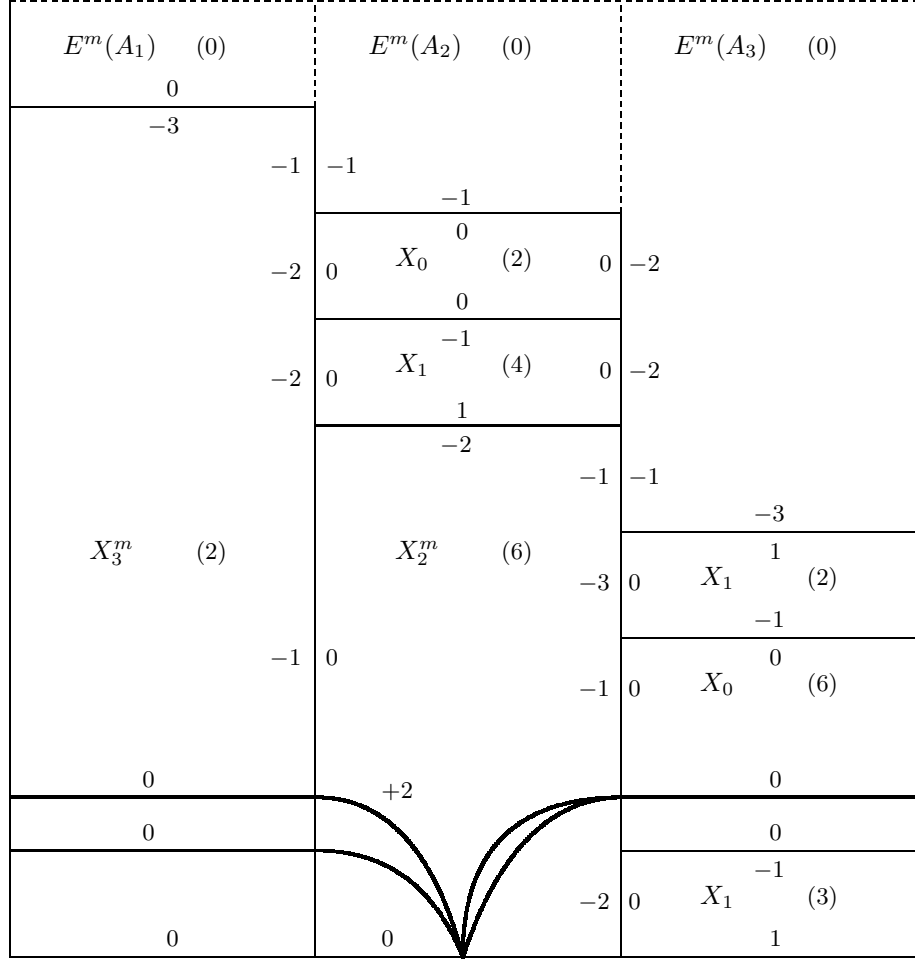
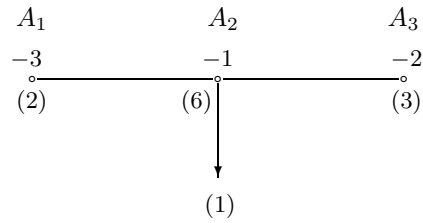
Let  $h : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$  be the germ of an analytic function and let  $(h)$  be the divisor of  $h \circ \tilde{\phi}$ . We assume that  $(h)$  is a normal crossing divisor. We write  $(h) = \sum_D m_D D$  where the sum runs over the irreducible exceptional divisors of  $\tilde{\phi}$  and the irreducible components of the strict transform of  $h$ . Let  $C$  be a compact curve determined by the intersection of two components  $D_1$  and  $D_2$ . Then

$$m_{D_1}(C^2 \text{ in } D_2) + m_{D_2}(C^2 \text{ in } D_1) + \sum m_D = 0$$

where the last sum is over the triple points  $D \cap D_1 \cap D_2$ . (For a proof, see e.g. [9].)

In our case, in order to obtain the self-intersections of  $S_i^m$ 's in  $St(g)$ , we apply the above relation for  $h = g = f + z^2$ .

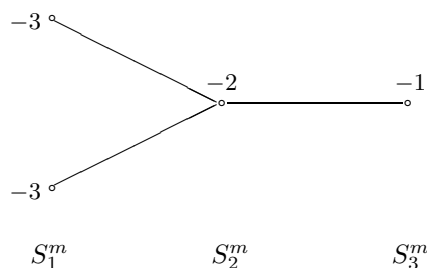
**3.8. Example.** Assume that  $f(x, y) = x^2 + y^3$ . We start with the following resolution of  $f$ .

FIGURE 7. The exceptional divisor, case  $x^2 + y^3 + z^2$ 

We fix the order  $A_1, A_2, A_3$  as it is indicated above. Then  $x_1 = 0, e_1 = -3$ ;  $x_2 = -1, e_2 = -1$ ;  $x_3 = -3, e_3 = -2$ . For the components of  $E$ , See Figure 7.

The dual resolution graph of  $\bigcup_i S_i^m \subset St(g)$  is





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